

AN EXPLICIT WALDSPURGER FORMULA FOR HILBERT MODULAR FORMS

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ABSTRACT. We describe a construction of preimages for the Shimura map on Hilbert modular forms, and give an explicit Waldspurger type formula relating their Fourier coefficients to central values of twisted L -functions. Our construction is inspired by that of Gross and applies to any nontrivial level and arbitrary base field, subject to certain conditions on the Atkin-Lehner eigenvalues and on the weight.

INTRODUCTION

Computing central values of L -functions attached to modular forms is interesting because of the arithmetic information they encode. These values are related to Fourier coefficients of half-integral weight modular forms and the Shimura correspondence, as shown in great generality in [Wal81]. For classical modular forms, explicit formulas of Waldspurger type can be found in [Gro87], [BSP90], among other works. In the Hilbert setting there are Waldspurger type formulas available in [Shi93], [BM07]. More explicit formulas for computing central values in terms of Fourier coefficients can be found in [HI13] in the case of trivial level, and in [Xue11] where the result is restricted to modular forms of prime power level over fields with odd class number. In [CST14] the authors give a formula in terms of heights, in the case of parallel weight $\mathbf{2}$.

In this article we prove a formula relating central values of twisted L -functions attached to a Hilbert cuspidal newform g to Fourier coefficients of certain modular forms of half-integral weight, which are constructed explicitly as theta series and map to g under the Shimura correspondence. Our result applies to any nontrivial level and arbitrary base field, and to a broader family of twists than the one considered in [Xue11]. In the classical case it is more general than [Gro87] and [BSP90], where the authors consider prime and square-free levels respectively.

Let g be a normalized Hilbert cuspidal newform over a totally real number field F , of level $\mathfrak{N} \subsetneq \mathcal{O}_F$, weight $\mathbf{2} + 2\mathbf{k}$ and trivial central character. For each $\mathfrak{p} \mid \mathfrak{N}$ denote by $\varepsilon_g(\mathfrak{p})$ the eigenvalue of the \mathfrak{p} -th Atkin-Lehner involution acting on g , and let $\mathcal{W}^- = \{\mathfrak{p} \mid \mathfrak{N} : \varepsilon_g(\mathfrak{p}) = -1\}$. We make the following hypotheses on \mathcal{W}^- and \mathbf{k} .

- H1.** $|\mathcal{W}^-|$ and $[F : \mathbb{Q}]$ have the same parity.
- H2.** $v_{\mathfrak{p}}(\mathfrak{N})$ is odd for every $\mathfrak{p} \in \mathcal{W}^-$.
- H3.** $(-1)^{\mathbf{k}} = 1$.

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For $D \in F^+$ denote by $L_D(s, g) = L(s, g)L(s, g \otimes \chi_D)$ the Rankin-Selberg convolution L -function of g by the quadratic character χ_D associated to the extension $F(\sqrt{-D})/F$, normalized with center of symmetry at $s = 1/2$. The main result of this article is stated in Theorem 5.3; in the simpler form given by Corollary 5.7 it claims that there exists a Hilbert cuspidal form f of weight $\mathbf{3}/2 + \mathbf{k}$ whose Fourier coefficients $\lambda(D, \mathfrak{a}; f)$ satisfy

$$L_D(1/2, g) = \frac{c_D}{D^{\mathbf{k}+1/2}} |\lambda(D, \mathfrak{a}; f)|^2,$$

under certain assumptions on D and \mathfrak{a} . Here c_D is an explicit positive rational number depending only on D . If $L(1/2, g) \neq 0$ then $f \neq 0$ and it maps to g under the Shimura correspondence. Actually, in this case we construct a linearly independent family of preimages for the Shimura correspondence, as shown in Corollary 5.6.

This article is organized as follows. In Section 1 we state some basic facts about the space of quaternionic modular forms. In Section 2 we show how to obtain half-integral weight Hilbert modular forms out of quaternionic modular forms, and give a formula for their Fourier coefficients in terms of special points and the height pairing on the space of quaternionic modular forms. In Section 3 we relate central values of twisted L -functions to the height pairing, using results of [Zha01] and [Xue06] about central values of Rankin L -functions. In Section 4 we state an auxiliary result, needed for the proof of the main result of this article, which we give in Section 5.

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Notation summary. We fix \mathbf{a} be a totally real number field F of discriminant d_F , with ring of integers \mathcal{O}_F . We denote by \mathcal{J}_F the group of fractional ideals of F , and we write $\text{Cl}(F)$ for the class group, and h_F for the class number. We denote by \mathbf{a} the set of embeddings $\tau : F \hookrightarrow \mathbb{R}$, and we let $F^+ = \{\xi \in F : \tau(\xi) > 0 \ \forall \tau \in \mathbf{a}\}$. Given $\mathbf{k} = (k_\tau) \in \mathbb{Z}^{\mathbf{a}}$ and $\xi \in F$, we let $\xi^{\mathbf{k}} = \prod_{\tau \in \mathbf{a}} \tau(\xi)^{k_\tau}$. By \mathfrak{p} we always denote a prime ideal of \mathcal{O}_F , and we use \mathfrak{p} as a subindex to denote completions of global objects at \mathfrak{p} . Given an integral ideal $\mathfrak{N} \subseteq \mathcal{O}_F$ we let $\omega(\mathfrak{N}) = |\{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{N}\}|$. Given \mathfrak{p} we denote by $\pi_{\mathfrak{p}}$ a local uniformizer at \mathfrak{p} , and we let $v_{\mathfrak{p}}$ denote the \mathfrak{p} -adic valuation.

Given a totally imaginary quadratic extension K/F we let \mathcal{O}_K be the maximal order, and let $\mathfrak{D}_K \subseteq \mathcal{O}_F$ denote the relative discriminant. We denote $t_K = [\mathcal{O}_K^\times : \mathcal{O}_F^\times]$, and we denote by $m_K \in \{1, 2\}$ the order of the kernel of the natural map $\text{Cl}(F) \rightarrow \text{Cl}(K)$.

Given a quaternion algebra B/F we denote by $\mathcal{N} : B^\times \rightarrow F^\times$ and $\mathcal{T} : B \rightarrow F$ the reduced norm and trace maps, and we use \mathcal{N} and \mathcal{T} to denote other norms and traces as well. We denote by $\widehat{B} = \prod'_{\mathfrak{p}} B_{\mathfrak{p}}$ and $\widehat{B}^\times = \prod'_{\mathfrak{p}} B_{\mathfrak{p}}^\times$ the corresponding restricted products, and we let $B_\infty = \prod_{\tau \in \mathbf{a}} B_\tau$. Finally, given a level $\mathfrak{N} \subseteq \mathcal{O}_F$, an integral or half-integral weight \mathbf{k} and a Hecke character χ , we denote by $\mathcal{M}_{\mathbf{k}}(\mathfrak{N}, \chi)$ and $\mathcal{S}_{\mathbf{k}}(\mathfrak{N}, \chi)$ the corresponding spaces of Hilbert modular and cuspidal forms.

1. QUATERNIONIC MODULAR FORMS

Let B be a totally definite quaternion algebra over F . Let (V, ρ) be an irreducible unitary right representation of B^\times/F^\times , which we denote by $(v, \gamma) \mapsto v \cdot \gamma$. Let R be an order of (reduced) discriminant \mathfrak{N} in B . A *quaternionic modular form* of weight

ρ and level R is a function $\varphi : \widehat{B}^\times \rightarrow V$ such that for every $x \in \widehat{B}^\times$ the following transformation formula is satisfied:

$$\varphi(ux\gamma) = \varphi(x) \cdot \gamma \quad \forall u \in \widehat{R}^\times, \gamma \in B^\times.$$

The space of all such functions is denoted by $\mathcal{M}_\rho(R)$. We let $\mathcal{E}_\rho(R)$ be the subspace of functions that factor through the map $\mathcal{N} : \widehat{B}^\times \rightarrow \widehat{F}^\times$. These spaces come equipped with the action of Hecke operators $T_{\mathfrak{m}}$, indexed by integral ideals $\mathfrak{m} \subseteq \mathcal{O}_F$, and given by

$$T_{\mathfrak{m}} \varphi(x) = \sum_{h \in \widehat{R}^\times \setminus H_{\mathfrak{m}}} \varphi(hx),$$

where $H_{\mathfrak{m}} = \{h \in \widehat{R} : \widehat{\mathcal{O}}_F \mathcal{N}(h) \cap \mathcal{O}_F = \mathfrak{m}\}$.

Given $x \in \widehat{B}^\times$, we let

$$\widehat{R}_x = x^{-1} \widehat{R} x, \quad R_x = B \cap \widehat{R}_x, \quad \Gamma_x = R_x^\times / \mathcal{O}_F^\times, \quad w_x = |\Gamma_x|.$$

The sets Γ_x are finite since B is totally definite. Let $\text{Cl}(R) = \widehat{R}^\times \setminus \widehat{B}^\times / B^\times$. We define an inner product on $\mathcal{M}_\rho(R)$, called the *height pairing*, by

$$\langle \varphi, \psi \rangle = \sum_{x \in \text{Cl}(R)} \frac{1}{w_x} \langle \varphi(x), \psi(x) \rangle.$$

The space of *cuspidal forms* $\mathcal{S}_\rho(R)$ is defined as the orthogonal complement of $\mathcal{E}_\rho(R)$ with respect to this pairing.

Let $N(\widehat{R}) = \{z \in \widehat{B}^\times : \widehat{R}_z = \widehat{R}\}$ be the normalizer of \widehat{R} in \widehat{B}^\times . We let $\widetilde{\text{Bil}}(R) = \widehat{R}^\times \setminus N(\widehat{R}) / F^\times$. We have an embedding $\text{Cl}(F) \hookrightarrow \widetilde{\text{Bil}}(R)$. The group $\widetilde{\text{Bil}}(R)$, and in particular $\text{Cl}(F)$, acts on $\mathcal{M}_\rho(R)$ by letting $(\varphi \cdot z)(x) = \varphi(zx)$. This action restricts to $\mathcal{S}_\rho(R)$, and is related to the height pairing by the equality

$$\langle \varphi \cdot z, \psi \cdot z \rangle = \langle \varphi, \psi \rangle.$$

The action of $\widetilde{\text{Bil}}(R)$ commutes with the action of the Hecke operators. The adjoint of $T_{\mathfrak{m}}$ with respect to the height pairing is given by $\varphi \mapsto T_{\mathfrak{m}} \varphi \cdot \mathfrak{m}^{-1}$.

The subspaces of $\mathcal{M}_\rho(R)$ and $\mathcal{S}_\rho(R)$ fixed by the action of $\text{Cl}(F)$ are denoted by $\mathcal{M}_\rho(R, \mathbb{1})$ and $\mathcal{S}_\rho(R, \mathbb{1})$. Let $\text{Bil}(R) = \widehat{R}^\times \setminus N(\widehat{R}) / \widehat{F}^\times$. Then $\text{Bil}(R)$ acts on $\mathcal{M}_\rho(R, \mathbb{1})$ and $\mathcal{S}_\rho(R, \mathbb{1})$, and we have a short exact sequence

$$(1.1) \quad 1 \longrightarrow \text{Cl}(F) \longrightarrow \widetilde{\text{Bil}}(R) \longrightarrow \text{Bil}(R) \longrightarrow 1.$$

Forms with minimal support. Given $x \in \widehat{B}^\times$ and $v \in V$, let $\varphi_{x,v} \in \mathcal{M}_\rho(R)$ be the quaternionic modular form given by

$$\varphi_{x,v}(y) = \sum_{\gamma \in \Gamma_{x,y}} v \cdot \gamma,$$

where $\Gamma_{x,y} = (B^\times \cap x^{-1} \widehat{R}^\times y) / \mathcal{O}_F^\times$. Note that $\varphi_{x,v}$ is supported in $\widehat{R}^\times x B^\times$. Furthermore, we have that

$$(1.2) \quad \varphi_{ux\gamma,v} = \varphi_{x,v \cdot \gamma^{-1}} \quad \forall u \in \widehat{R}^\times, \gamma \in B^\times,$$

$$(1.3) \quad \varphi_{x,v} \cdot z = \varphi_{z^{-1}x,v} \quad \forall z \in \widetilde{\text{Bil}}(R).$$

Given $\varphi \in \mathcal{M}_\rho(R)$, using that $\varphi(x) \in V^{\Gamma_x}$ for every $x \in \widehat{B}^\times$ we get that

$$(1.4) \quad \varphi = \sum_{x \in \text{Cl}(R)} \frac{1}{w_x} \varphi_{x, \varphi(x)}.$$

Proposition 1.5. *Let $x \in \widehat{B}^\times$ and $v \in V$. Then $T_{\mathfrak{m}} \varphi_{x,v} = \sum_{h \in H_{\mathfrak{m}}/\widehat{R}^\times} \varphi_{h^{-1}x,v}$.*

Proof. Given $y \in \widehat{B}^\times$, let $\Gamma = (B^\times \cap x^{-1}H_{\mathfrak{m}}y)/\mathcal{O}_F^\times$. Then

$$\Gamma = \coprod_{h \in \widehat{R}^\times \setminus H_{\mathfrak{m}}} \Gamma_{x,hy} = \coprod_{h \in H_{\mathfrak{m}}/\widehat{R}^\times} \Gamma_{h^{-1}x,y}.$$

The first decomposition implies that

$$T_{\mathfrak{m}} \varphi_{x,v}(y) = \sum_{h \in \widehat{R}^\times \setminus H_{\mathfrak{m}}} \varphi_{x,v}(hy) = \sum_{h \in \widehat{R}^\times \setminus H_{\mathfrak{m}}} \sum_{\gamma \in \Gamma_{x,hy}} v \cdot \gamma = \sum_{\beta \in \Gamma} v \cdot \beta,$$

whereas the second decomposition implies that

$$\sum_{h \in H_{\mathfrak{m}}/\widehat{R}^\times} \varphi_{h^{-1}x,v}(y) = \sum_{h \in H_{\mathfrak{m}}/\widehat{R}^\times} \sum_{\gamma \in \Gamma_{h^{-1}x,y}} v \cdot \gamma = \sum_{\beta \in \Gamma} v \cdot \beta,$$

which completes the proof. \square

Proposition 1.6. *Given $x, y \in \widehat{B}^\times$ and $v, w \in V$, we have*

$$\langle \varphi_{x,v}, \varphi_{y,w} \rangle = \sum_{\gamma \in \Gamma_{x,y}} \langle v \cdot \gamma, w \rangle.$$

Proof. Since $\Gamma_y = \Gamma_{y,y}$ acts on $\Gamma_{x,y}$ on the right, using that (ρ, V) is unitary we get that

$$\begin{aligned} \langle \varphi_{x,v}, \varphi_{y,w} \rangle &= \frac{1}{w_y} \langle \varphi_{x,v}(y), \varphi_{y,w}(y) \rangle = \frac{1}{w_y} \sum_{\alpha \in \Gamma_{x,y}} \sum_{\beta \in \Gamma_y} \langle v \cdot \alpha, w \cdot \beta \rangle \\ &= \frac{1}{w_y} \sum_{\alpha \in \Gamma_{x,y}} \sum_{\beta \in \Gamma_y} \langle v \cdot \alpha \beta^{-1}, w \rangle = \sum_{\gamma \in \Gamma_{x,y}} \langle v \cdot \gamma, w \rangle. \end{aligned} \quad \square$$

2. HALF-INTEGRAL WEIGHT MODULAR FORMS AND SPECIAL POINTS

From now on we specify the representation (V, ρ) . Let $\mathbf{k} = (k_\tau) \in \mathbb{Z}_{\geq 0}^{\mathbf{a}}$. For each $\tau \in \mathbf{a}$ we consider the real vector space $W_\tau = B_\tau/F_\tau$, with inner product induced by the totally positive definite quadratic form $-\Delta(x) = 4\mathcal{N}(x) - \mathcal{T}(x)^2$. By letting $B_\tau^\times/F_\tau^\times$ act on W_τ by conjugation we get an orthogonal representation. Then $B_\tau^\times/F_\tau^\times$ acts on $\text{Sym}^{k_\tau}(\text{Hom}_{\mathbb{R}}(W_\tau, \mathbb{C}))$, the space of homogeneous polynomials on W_τ of degree k_τ with coefficients in \mathbb{C} . We let V_{k_τ} denote the $B_\tau^\times/F_\tau^\times$ -submodule of harmonic polynomials with respect to $-\Delta$. This is, up to isomorphism, the unique irreducible unitary representation of $B_\tau^\times/F_\tau^\times$ of dimension $2k_\tau + 1$. We let $V_{\mathbf{k}} = \otimes_{\tau \in \mathbf{a}} V_{k_\tau}$, and through the embedding $B^\times \hookrightarrow B_\infty^\times$ we get an irreducible unitary representation $(V_{\mathbf{k}}, \rho_{\mathbf{k}})$ of B^\times/F^\times . We denote the corresponding spaces of quaternionic modular forms by $\mathcal{M}_{\mathbf{k}}(R)$, etc.

Denote by \mathcal{H} the complex upper half-plane. Let $e_F : F \times \mathcal{H}^{\mathbf{a}} \rightarrow \mathbb{C}$ be the exponential function given by $e_F(\xi, z) = \exp(2\pi i \sum_{\tau \in \mathbf{a}} \tau(\xi) z_\tau)$. Given $x \in \widehat{B}^\times$, let $L_x \subseteq B/F$ be

the lattice given by $L_x = R_x/\mathcal{O}_F$. Given $P \in V_{\mathbf{k}}$, we let $\vartheta_{x,P} : \mathcal{H}^{\mathbf{a}} \rightarrow \mathbb{C}$ be the function given by

$$\vartheta_{x,P}(z) = \sum_{y \in L_x} P(y) e_F(-\Delta(y), z/2).$$

These theta series satisfy

$$(2.1) \quad \vartheta_{x,P} = (-1)^{\mathbf{k}} \vartheta_{x,P},$$

$$(2.2) \quad \vartheta_{zx\gamma, P \cdot \gamma} = \vartheta_{x,P}, \quad \forall z \in N(\widehat{R}), \gamma \in B^\times.$$

The following two propositions extend the results about theta series from [Sir14, Section 4] to arbitrary weights.

Proposition 2.3. *Let $x \in \widehat{B}^\times$ and let $P \in V_{\mathbf{k}}$. Then $\vartheta_{x,P} \in \mathcal{M}_{\mathbf{3}/2+\mathbf{k}}(4\mathfrak{N}, \chi_1)$, where χ_1 is the Hecke character associated to the extension $F(\sqrt{-1})/F$. Furthermore, for $D \in F^+ \cup \{0\}$ and $\mathfrak{a} \in \mathcal{J}_F$, the D -th Fourier coefficient of $\vartheta_{x,P}$ at the cusp \mathfrak{a} is given by*

$$(2.4) \quad \lambda(D, \mathfrak{a}; \vartheta_{x,P}) = \frac{1}{\mathcal{N}(\mathfrak{a})} \sum_{y \in \mathcal{A}_{D,\mathfrak{a}}(x)} P(y),$$

where $\mathcal{A}_{D,\mathfrak{a}}(x) = \{y \in \mathfrak{a}^{-1}L_x : \Delta(y) = -D\}$.

Proof. Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be a basis of B/F such that there exist $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \in \mathcal{J}_F$ satisfying $L_x = \bigoplus_{i=1}^3 \mathfrak{a}_i v_i$. Let $S \in \mathrm{GL}_3(F)$ be the matrix of the quadratic form Δ with respect to this basis. For each $\tau \in \mathfrak{a}$ write $\tau(S) = -A_\tau^t A_\tau$, with $A_\tau \in \mathrm{GL}_3(\mathbb{R})$. Then there exist homogeneous harmonic polynomials $Q_\tau(X)$ of degree k_τ such that

$$P(y) = \prod_{\tau \in \mathfrak{a}} Q_\tau(A_\tau \tau([y]_{\mathcal{B}})),$$

where we denote by $[y]_{\mathcal{B}}$ the coordinates of y with respect to the basis \mathcal{B} . For each τ we may assume that $Q_\tau(X) = (z_\tau^t X)^{k_\tau}$, with $z_\tau \in \mathbb{C}^3$ such that $z_\tau^t z_\tau = 0$ (see [Iwa97, Theorem 9.1]). Let $\rho_\tau = -(A_\tau^{-1} z_\tau)^t$. Then we have that $\rho_\tau^t \tau(S) \rho_\tau = 0$. Let $\sigma : F^3 \rightarrow \mathbb{C}$ be the function given by $\sigma(\xi) = \prod_{\tau \in \mathfrak{a}} (\rho_\tau^t \tau(S) \tau(\xi))^{k_\tau}$, and let $\eta : F^3 \rightarrow \mathbb{C}$ be the characteristic function of $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \mathfrak{a}_3$. Then we have that $\vartheta_{x,P} = \overline{f}$, where f is the function given by

$$f(z) = \sum_{\xi \in F^3} \eta(\xi) \sigma(\xi) e_F(\xi^t S \xi, \overline{z}/2).$$

The modularity of $\vartheta_{x,P}$ follows by applying [Shi87, Proposition 11.8] to the function f . Finally, (2.4) follows as in the case $\mathbf{k} = \mathbf{0}$ considered in [Sir14, Proposition 4.4]. \square

We use the theta series $\vartheta_{x,P}$ to define a linear map $\vartheta : \mathcal{M}_{\mathbf{k}}(R) \rightarrow \mathcal{M}_{\mathbf{3}/2+\mathbf{k}}(4\mathfrak{N}, \chi_1)$, given by $\vartheta(\varphi_{x,P}) = \vartheta_{x,P}$. This map is well defined by (1.4) and (2.2), and satisfies

$$\vartheta(\varphi) = \sum_{x \in \mathrm{Cl}(R)} \frac{1}{w_x} \vartheta_{x,\varphi(x)}.$$

Note that if Hypothesis H3 does not hold, then $\vartheta = 0$ by (2.1).

Proposition 2.5. *The map ϑ is Hecke-linear, and satisfies that*

$$\vartheta(\varphi \cdot z) = \vartheta(\varphi) \quad \forall z \in \widetilde{\text{Bil}}(R).$$

Furthermore, $\vartheta(\varphi)$ is cuspidal if and only if φ is cuspidal.

Proof. For $z \in \widetilde{\text{Bil}}(R)$ we have, by (1.3) and (2.2), that

$$\vartheta(\varphi_{x,P} \cdot z) = \vartheta(\varphi_{z^{-1}x,P}) = \vartheta(\varphi_{x,P}).$$

The Hecke-linearity was proved in [Sir14, Theorem 4.11] when $\mathbf{k} = \mathbf{0}$, and can be proved in this setting following the same lines. The assertion on the cuspidality was proved in [Sir14, Theorem 4.11] when $\mathbf{k} = \mathbf{0}$, and in the remaining cases follows from the facts that $\mathcal{M}_{\mathbf{k}}(R) = \mathcal{S}_{\mathbf{k}}(R)$ and that every theta series is cuspidal. \square

From now on D denotes an element in F^+ , and we denote $K = F(\sqrt{-D})$. Assume that there exists an embedding $K \hookrightarrow B$, which we fix. Let $P_D \in V_{\mathbf{k}}$ be the polynomial characterized by the property

$$(2.6) \quad P(\omega) = \langle P, P_D \rangle \quad \forall P \in V_{\mathbf{k}},$$

where $\omega \in K/F$ is such that $\Delta(\omega) = -D$. Note that ω is uniquely determined up to sign. By Hypothesis H3 we have $P(-\omega) = P(\omega)$ for every $P \in V_{\mathbf{k}}$, which implies that P_D does not depend on ω . Since $(P \cdot a)(\omega) = P(\omega)$ for every $a \in K^\times$, we have that

$$(2.7) \quad P_D \cdot a = P_D \quad \forall a \in K^\times / F^\times.$$

Proposition 2.8. *The polynomial P_D satisfies*

$$(2.9) \quad \langle P_D, P_D \rangle = D^{\mathbf{k}} \prod_{\tau \in \mathbf{a}} s_{k_\tau},$$

where for $k \in \mathbb{Z}_{\geq 0}$ we denote by s_k the positive rational number given by

$$(2.10) \quad s_k = \frac{1}{\Gamma(k+1/2)} \sum_{q=0}^{\lfloor \frac{k}{2} \rfloor} \frac{\Gamma(k+1/2-q)}{q! (k-2q)! 2^{2q}}.$$

Proof. We have that $P_D = \otimes_{\tau \in \mathbf{a}} P_{D,\tau}$, where $P_{D,\tau} \in V_{k_\tau}$ is the polynomial characterized by the property

$$P_\tau(\omega) = \langle P_\tau, P_{D,\tau} \rangle \quad \forall P_\tau \in V_{k_\tau}.$$

Identifying B_τ with Hamilton quaternions $\langle 1, i, j, ij \rangle_{\mathbb{R}}$ and letting $X_1 = i/2$, $X_2 = j/2$, $X_3 = ij/2$, we have that $\{X_1, X_2, X_3\}$ is an orthonormal basis for W_τ with respect to $-\Delta$. Then the monomials $X_1^a X_2^b X_3^{k_\tau-a-b} \in \text{Sym}^{k_\tau}(\text{Hom}_{\mathbb{R}}(W_\tau, \mathbb{C}))$ are orthogonal, and have norm equal to $a!b!(k_\tau-a-b)!$, which implies that the inner product $\langle \cdot, \cdot \rangle$ we consider on V_{k_τ} is related to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ considered in [Geb09, Section 4.1] by $\langle \cdot, \cdot \rangle = k_\tau! \langle\langle \cdot, \cdot \rangle\rangle$. Hence (2.9) follows from the explicit formulas for the Gegenbauer polynomials given in [Geb09, Proposition 4.1.9], which imply that

$$\langle\langle P_{D,\tau}, P_{D,\tau} \rangle\rangle = \frac{\tau(D)^{k_\tau} s_{k_\tau}}{k_\tau!}.$$

\square

Given $\mathfrak{a} \in \mathcal{J}_F$, we say that the pair $(-D, \mathfrak{a})$ is a *discriminant* if there exists $\omega \in K$ with $\Delta(\omega) = -D$ such that $\mathcal{O}_F \oplus \mathfrak{a}\omega$ is an order in K . In this case it is the unique order in K of discriminant $D\mathfrak{a}^2$, and in particular it does not depend on ω . We denote it by $\mathcal{O}_{D,\mathfrak{a}}$. We say that the discriminant $(-D, \mathfrak{a})$ is *fundamental* if $\mathcal{O}_{D,\mathfrak{a}} = \mathcal{O}_K$.

Proposition 2.11. *Let \mathcal{O} be an \mathcal{O}_F -order in K . Then there exists a unique $\mathfrak{a} \in \mathcal{J}_F$ such that $(-D, \mathfrak{a})$ is a discriminant with $\mathcal{O}_{D,\mathfrak{a}} = \mathcal{O}$.*

Proof. Let r be an \mathcal{O}_F -linear retraction for the embedding $\mathcal{O}_F \hookrightarrow \mathcal{O}$, which we extend to a F -linear map $r : K \rightarrow F$. Let $\omega' \in K$ be any element satisfying $\Delta(\omega') = -D$, and let $\omega = \omega' - r(\omega')$. Then, since $\omega \notin F$, we have that $\ker r = F\omega$. Hence letting $\mathfrak{a} = (\mathcal{O}\omega^{-1}) \cap F$ we have that $\mathcal{O} = \mathcal{O}_F \oplus \mathfrak{a}\omega$. Note that the ideal \mathfrak{a} is uniquely determined since $D\mathfrak{a}^2$ is the discriminant of \mathcal{O} . \square

Proposition 2.12. *Let \mathfrak{p} be a prime ideal. If $(-D, \mathfrak{a})$ is a discriminant then $(-D, \mathfrak{p}\mathfrak{a})$ is a discriminant, and the converse is true if $\mathfrak{p} \nmid 2$ and $D \in \mathfrak{a}^{-2}$.*

Proof. The first statement is trivial. To prove the converse, assume that $\mathcal{O}_F \oplus \mathfrak{p}\mathfrak{a}\omega$ is an order in K , with $\Delta(\omega) = -D$. In particular, we have that $\mathcal{T}(\omega) \in (\mathfrak{p}\mathfrak{a})^{-1}$ and $\mathcal{N}(\omega) \in (\mathfrak{p}\mathfrak{a})^{-2}$. Since $\mathfrak{p} \nmid 2$, there exists $\xi \in \mathcal{O}_F$ such that $1 - 2\xi \in \mathfrak{p}$. Then, changing ω by $\omega - \xi\mathcal{T}(\omega)$, we may assume that $\mathcal{T}(\omega) \in \mathfrak{a}^{-1}$.

By hypothesis we have that $\Delta(\omega) = \mathcal{T}(\omega)^2 - 4\mathcal{N}(\omega) \in \mathfrak{a}^{-2}$. In particular, since $\mathcal{T}(\omega) \in \mathfrak{a}^{-1}$ we have that $4\mathcal{N}(\omega) \in \mathfrak{a}^{-2}$. Since $\mathfrak{p} \nmid 2$ and $\mathcal{N}(\omega) \in (\mathfrak{p}\mathfrak{a})^{-2}$ we have that $\mathcal{N}(\omega) \in \mathfrak{a}^{-2}$, which allows us to conclude that $\mathcal{O}_F \oplus \mathfrak{a}\omega$ is an order in K . \square

Let $(-D, \mathfrak{a})$ be a discriminant, and let $\tilde{X}_{D,\mathfrak{a}} = \{x \in \hat{B}^\times : \mathcal{O}_{D,\mathfrak{a}} \subseteq R_x\}$. We define a set $X_{D,\mathfrak{a}}$ of *special points* associated to the discriminant $(-D, \mathfrak{a})$ by

$$X_{D,\mathfrak{a}} = \hat{R}^\times \backslash \tilde{X}_{D,\mathfrak{a}} / K^\times.$$

If $(-D, \mathfrak{a})$ is not a discriminant, we let $X_{D,\mathfrak{a}} = \emptyset$. Let

$$\eta_{D,\mathfrak{a}} = \sum_{x \in X_{D,\mathfrak{a}}} \frac{1}{[\mathcal{O}_x^\times : \mathcal{O}_F^\times]} \varphi_{x,P_D} \in \mathcal{M}_{\mathbf{k}}(R),$$

where $\mathcal{O}_x = R_x \cap K$. This is well defined by (1.2) and (2.7). When $(-D, \mathfrak{a})$ is fundamental then

$$(2.13) \quad \eta_{D,\mathfrak{a}} = \frac{1}{t_K} \sum_{x \in X_{D,\mathfrak{a}}} \varphi_{x,P_D},$$

because in this case $\mathcal{O}_x = \mathcal{O}_K$ for every $x \in X_{D,\mathfrak{a}}$. It can be proved that $\eta_{D,\mathfrak{a}}$ does not depend on the choice of the embedding $K \hookrightarrow B$. When there does not exist such an embedding, we let $\eta_{D,\mathfrak{a}} = 0$.

Proposition 2.14. *Let $\varphi \in \mathcal{M}_{\mathbf{k}}(R)$. Let $D \in F^+$ and let $\mathfrak{a} \in \mathcal{J}_F$. Then the D -th Fourier coefficient of $\vartheta(\varphi)$ at the cusp \mathfrak{a} is given by*

$$(2.15) \quad \lambda(D, \mathfrak{a}; \vartheta(\varphi)) = \frac{1}{\mathcal{N}(\mathfrak{a})} \langle \varphi, \eta_{D,\mathfrak{a}} \rangle.$$

Proof. By (1.4) we can assume that $\varphi = \varphi_{x,P}$ with $P \in V_{\mathbf{k}}^{\Gamma_x}$, so that $\vartheta(\varphi) = \vartheta_{x,P}$. If K does not embed into B , then $\mathcal{A}_{D,\mathfrak{a}}(x) = \emptyset$ and both sides of (2.15) vanish.

Fix $\omega \in K$ with $\Delta(\omega) = -D$, and let Γ_x act on $\mathcal{A}_{D,\mathfrak{a}}(x)$ by conjugation. Given $y \in \mathcal{A}_{D,\mathfrak{a}}(x)$, since $\Delta(y) = \Delta(\omega)$, we can assume there exists $\gamma \in B^\times$ such that $y = \gamma\omega\gamma^{-1}$. In particular $\mathcal{O}_{D,\mathfrak{a}} = \mathcal{O}_F \oplus \mathfrak{a}\omega$. The map $y \mapsto x\gamma$ induces an injection

$$\Gamma_x \backslash \mathcal{A}_{D,\mathfrak{a}}(x) \longrightarrow X_{D,\mathfrak{a}}.$$

Note that $\text{Stab}_{\Gamma_x} y = (R_x \cap F(y))/\mathcal{O}_F^\times \simeq (R_{x\gamma} \cap K)/\mathcal{O}_F^\times = \mathcal{O}_{x\gamma}^\times/\mathcal{O}_F^\times$. Note also that $P(y) = \langle P \cdot \gamma, P_D \rangle = \frac{1}{w_x} \langle \varphi_{x,P}, \varphi_{x\gamma, P_D} \rangle$, using that P is fixed by Γ_x . Hence

$$\begin{aligned} \mathcal{N}(\mathfrak{a}) \lambda(D, \mathfrak{a}; \vartheta_{x,P}) &= \sum_{y \in \mathcal{A}_{D,\mathfrak{a}}(x)} P(y) = \sum_{y \in \Gamma_x \backslash \mathcal{A}_{D,\mathfrak{a}}(x)} [\Gamma_x : \text{Stab}_{\Gamma_x} y] P(y) \\ &= \sum_{x\gamma \in X_{D,\mathfrak{a}}} \frac{w_x}{[\mathcal{O}_{x\gamma}^\times : \mathcal{O}_F^\times]} \langle P \cdot \gamma, P_D \rangle \\ &= \sum_{z \in X_{D,\mathfrak{a}}} \frac{1}{[\mathcal{O}_z^\times : \mathcal{O}_F^\times]} \langle \varphi_{x,P}, \varphi_{z, P_D} \rangle = \langle \varphi_{x,P}, \eta_{D,\mathfrak{a}} \rangle. \end{aligned}$$

Note that in the last sum $\langle \varphi_{x,P}, \varphi_{z, P_D} \rangle = 0$ unless $z = x\gamma$. \square

By analogy with the case $F = \mathbb{Q}$ (see [Koh82]), we consider the *plus subspace* of $\mathcal{M}_{\mathbf{3}/2+\mathbf{k}}(4\mathfrak{N}, \chi_1)$ which, under Hypothesis H3, is given by

$$\begin{aligned} \mathcal{M}_{\mathbf{3}/2+\mathbf{k}}^+(4\mathfrak{N}, \chi_1) &= \{f \in \mathcal{M}_{\mathbf{3}/2+\mathbf{k}}(4\mathfrak{N}, \chi_1) : \\ &\quad \lambda(D, \mathfrak{a}; f) = 0 \text{ unless } (-D, \mathfrak{a}) \text{ is a discriminant}\}, \end{aligned}$$

and we let $\mathcal{S}_{\mathbf{3}/2+\mathbf{k}}^+(4\mathfrak{N}, \chi_1) = \mathcal{M}_{\mathbf{3}/2+\mathbf{k}}^+(4\mathfrak{N}, \chi_1) \cap \mathcal{S}_{\mathbf{3}/2+\mathbf{k}}(4\mathfrak{N}, \chi_1)$. Using the formula for the action of the Hecke operators in terms of Fourier coefficients (see [Shi87, Proposition 5.4]) together with Proposition 2.12 it is easy to prove that $\mathcal{M}_{\mathbf{3}/2+\mathbf{k}}^+(4\mathfrak{N}, \chi_1)$ is stable by the Hecke operators $T_{\mathfrak{p}}$ with $\mathfrak{p} \nmid 2$.

Corollary 2.16. *The Hecke-linear map ϑ sends $\mathcal{M}_{\mathbf{k}}(R)$ into $\mathcal{M}_{\mathbf{3}/2+\mathbf{k}}^+(4\mathfrak{N}, \chi_1)$.*

3. HEIGHT AND GEOMETRIC PAIRINGS

We start this section by comparing the geometric pairing on CM-cycles of [Zha01] (see [Xue06] for the case of higher weight) with the height pairing introduced in Section 1.

Let K/F be a totally imaginary quadratic extension. As in Section 2 we assume that there exists an embedding $K \hookrightarrow B$, which we fix. Furthermore, we assume $\mathcal{O}_K \subseteq R$. Let $\mathcal{C} = (\widehat{B}^\times / \widehat{F}^\times) / (K^\times / F^\times)$, and let $\pi : \widehat{B}^\times / \widehat{F}^\times \rightarrow \mathcal{C}$ denote the projection map. We fix a Haar measure μ on $\widehat{B}^\times / \widehat{F}^\times$. On K^\times / F^\times we consider the discrete measure, and we let $\bar{\mu}$ be the quotient measure on \mathcal{C} . We write $\mu_R = \mu(\widehat{R}^\times / \widehat{F}^\times)$.

We consider the space $\mathcal{D}(\mathcal{C})$ of CM-cycles on \mathcal{C} . These are locally constant functions on \mathcal{C} with compact support. This space comes equipped with the action of Hecke operators $T_{\mathfrak{m}}$ given by

$$(3.1) \quad T_{\mathfrak{m}} \alpha(x) = \frac{1}{\mu_R} \int_{H_{\mathfrak{m}} / \widehat{\mathcal{O}}_F^\times} \alpha(hx) dh.$$

Given $v \in V$ which is fixed by K^\times/F^\times , we let $M_v : B^\times/F^\times \rightarrow \mathbb{C}$ be the matrix coefficient given by $\gamma \mapsto \langle v \cdot \gamma, v \rangle$. Then M_v is bi- K^\times/F^\times -invariant and satisfies that $\overline{M_v(\gamma)} = M_v(\gamma^{-1})$. We call M_v a *multiplicity function*. We let $k_v : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{C}$ be the map given by

$$k_v(x, y) = \sum_{\gamma \in \Gamma'_{x,y}} M_v(\gamma),$$

where for $x, y \in \widehat{B}^\times$ we denote $\Gamma'_{x,y} = (B^\times \cap x^{-1} \widehat{F}^\times \widehat{R}^\times y) / F^\times$. We consider the *geometric pairing* on $\mathcal{D}(\mathcal{C})$ induced by M_v , which for $\alpha, \beta \in \mathcal{D}(\mathcal{C})$ that are left invariant by $\widehat{R}^\times / \widehat{\mathcal{O}}_F^\times$ is given by

$$(3.2) \quad \langle \alpha, \beta \rangle_v = \frac{1}{\mu_R} \iint_{\mathcal{C} \times \mathcal{C}} \alpha(x) \overline{\beta(y)} k_v(x, y) dx dy.$$

Lemma 3.3. *Let $x, y \in \widehat{B}^\times$. The natural map $\Gamma_{x,y} \rightarrow \Gamma'_{x,y}$ is injective. Furthermore*

$$\Gamma'_{x,y} = \coprod_{\xi \in \text{Cl}(F)} \Gamma_{\xi x, y}.$$

Proof. Let $u, v \in \widehat{R}^\times$ be such that there exists $\eta \in F^\times$ with $x^{-1} u y \eta = x^{-1} v y$. Then $\eta = u^{-1} v \in F^\times \cap \widehat{R}^\times = \mathcal{O}_F^\times$. This proves the first statement.

It is clear that the union gives all of $\Gamma'_{x,y}$. To see that it is disjoint, suppose that $\xi, \zeta \in \widehat{F}^\times$ are such that there exist $u, v \in \widehat{R}^\times$ and $\eta \in F^\times$ with $x^{-1} \xi u y \eta = x^{-1} \zeta v y$. Then $\xi \zeta^{-1} \eta = v u^{-1} \in \widehat{F}^\times \cap \widehat{R}^\times = \mathcal{O}_F^\times$, and hence $\xi = \zeta$ in $\text{Cl}(F)$. \square

The following result is immediate from this lemma and Proposition 1.6.

Proposition 3.4. *Let $x, y \in \widehat{B}^\times$. Then $k_v(x, y) = \sum_{\xi \in \text{Cl}(F)} \langle \varphi_{\xi x, v}, \varphi_{y, v} \rangle$.*

Given $a \in \widehat{K}^\times$, we let $\alpha_a \in \mathcal{D}(\mathcal{C})$ be the characteristic function of $\pi(\widehat{R}^\times a) \subseteq \mathcal{C}$. Since $\mathcal{O}_K \subseteq R$, the CM-cycle α_a depends only on the element in $\text{Cl}(K)$ determined by a . The same holds for the quaternionic modular form $\varphi_{a,v}$ by (1.2).

Proposition 3.5. *Let $\mathfrak{m} \subseteq \mathcal{O}_F$ be an ideal. For $a, b \in \text{Cl}(K)$ we have that*

$$\frac{\langle T_{\mathfrak{m}} \alpha_a, \alpha_b \rangle_v}{\mu_R} = \frac{1}{t_K^2} \sum_{\xi \in \text{Cl}(F)} \langle T_{\mathfrak{m}} \varphi_{\xi a, v}, \varphi_{b, v} \rangle.$$

Proof. Using (3.1) and (3.2), we obtain that

$$\begin{aligned} \frac{\langle T_{\mathfrak{m}} \alpha_a, \alpha_b \rangle_v}{\mu_R} &= \frac{1}{\mu_R^3} \iint_{\mathcal{C} \times \mathcal{C}} \int_{H_{\mathfrak{m}} / \widehat{\mathcal{O}}_F^\times} \alpha_a(hx) \alpha_b(y) k_v(x, y) dh dx dy \\ &= \frac{1}{\mu_R^3} \iint_{\pi(\widehat{R}^\times a) \times \pi(\widehat{R}^\times b)} \int_{H_{\mathfrak{m}} / \widehat{\mathcal{O}}_F^\times} k_v(h^{-1}x, y) dh dx dy. \end{aligned}$$

Note that

$$\frac{1}{\mu_R} \int_{H_{\mathfrak{m}} / \widehat{\mathcal{O}}_F^\times} k_v(h^{-1}x, y) dh = \sum_{h \in H_{\mathfrak{m}} / \widehat{R}^\times} k_v(h^{-1}x, y)$$

is constant on $\pi(\widehat{R}^\times a) \times \pi(\widehat{R}^\times b)$. Furthermore, $\mu_R / \mu(\pi(\widehat{R}^\times)) = |K^\times / F^\times \cap \widehat{R}^\times / \widehat{\mathcal{O}}_F^\times| = t_K$.

Using this and Proposition 3.4 we get that

$$\frac{\langle T_{\mathfrak{m}} \alpha_a, \alpha_b \rangle_v}{\mu_R} = \frac{1}{t_K^2} \sum_{h \in H_{\mathfrak{m}} / \widehat{R}^\times} k_v(h^{-1}a, b) = \frac{1}{t_K^2} \sum_{\xi \in \text{Cl}(F)} \sum_{h \in H_{\mathfrak{m}} / \widehat{R}^\times} \langle \varphi_{h^{-1}\xi a, v}, \varphi_{b, v} \rangle.$$

Then the result follows from Proposition 1.5. \square

Let $\alpha_K \in \mathcal{D}(\mathcal{C})$ be the characteristic function of $\pi(\widehat{R}^\times \widehat{K}^\times)$. We have that

$$\alpha_K = \frac{m_K}{h_F} \sum_{a \in \text{Cl}(K)} \alpha_a.$$

Similarly we define

$$(3.6) \quad \psi_v = \frac{1}{t_K} \sum_{a \in \text{Cl}(K)} \varphi_{a, v} \in \mathcal{M}_\rho(R, \mathbb{1}).$$

After these definitions and Proposition 3.5 we get the following result, analogous to [Xue06, Corollary 3.5], where the author only considers the case when $F = \mathbb{Q}$ and \mathfrak{N} is square-free.

Corollary 3.7. *Let $\mathfrak{m} \subseteq \mathcal{O}_F$ be an ideal. Then*

$$\frac{\langle T_{\mathfrak{m}} \alpha_K, \alpha_K \rangle_v}{\mu_R} = \frac{m_K^2}{h_F} \langle T_{\mathfrak{m}} \psi_v, \psi_v \rangle.$$

Central values. Let g be a normalized Hilbert cuspidal newform over F of level \mathfrak{N} and trivial central character as in the introduction. For the rest of this section we specify the quaternion algebra B , letting it ramify exactly at the archimedean primes and at \mathcal{W}^- , which is possible due to Hypothesis H1. Let T_g be a polynomial in the Hecke operators prime to \mathfrak{N} giving the g -isotypical projection.

The following result is [Xue06, Theorem 1.2], which was originally proved for parallel weight 2 in [Zha01].

Theorem 3.8. *Assume $\mathfrak{N} \subsetneq \mathcal{O}_F$ and \mathfrak{D}_K is prime to $2\mathfrak{N}$. Then*

$$(3.9) \quad L_D(1/2, g) = \langle g, g \rangle \frac{d_F^{1/2} C(\mathfrak{N})}{\mathcal{N}(\mathfrak{D}_K)^{1/2}} \frac{\langle\langle T_g \alpha_K, \alpha_K \rangle\rangle}{\mu_R},$$

where $C(\mathfrak{N})$ is the positive rational constant given by

$$C(\mathfrak{N}) = \prod_{\mathfrak{p} | \mathfrak{N}} (\mathcal{N}(\mathfrak{p}) + 1) \mathcal{N}(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{N}) - 1},$$

and where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the geometric pairing in $\mathcal{D}(\mathcal{C})$ given in [Xue06, (3.4)].

Remark 3.10. The constant C_1 mentioned in [Xue06, Theorem 1.2] contains a wrong factor, so we refer to [Xue06, (3.65)], The constants $\mu_{\mathfrak{N} \mathfrak{D}_K}$, μ_{Δ^*} and μ_{Δ} appearing in the latter satisfy

$$\mu_{\mathfrak{N} \mathfrak{D}_K}^{-1} = C(\mathfrak{N} \mathfrak{D}_K) = C(\mathfrak{N}) C(\mathfrak{D}_K), \quad \mu_{\Delta^*} = C(\mathfrak{D}_K) \mu_{\Delta} = 2^{|S|} \mu_R.$$

Using this we obtain (3.9).

Remark 3.11. The proof given in [Xue06] is valid for a particular order in B containing \mathcal{O}_K . Since by [Gro88, Proposition 3.4] any two orders in B containing \mathcal{O}_K are locally conjugate by an element of \widehat{K}^\times , and the right hand side of (3.9) is invariant by such a conjugation, Theorem 3.8 holds for any order in B containing \mathcal{O}_K .

Corollary 3.12. *Under the hypotheses above, assume that $V = V_{\mathbf{k}}$ as in Section 2, and let $P_D \in V_{\mathbf{k}}$ as in (2.6). Then*

$$L_D(1/2, g) = \langle g, g \rangle \frac{d_F^{1/2}}{h_F} \frac{c(\mathbf{k}) C(\mathfrak{N})}{\mathcal{N}(\mathfrak{D}_K)^{1/2}} \frac{m_K^2}{D^{\mathbf{k}}} \langle T_g \psi_{P_D}, \psi_{P_D} \rangle.$$

Here $c(\mathbf{k})$ is the positive rational constant given by $c(\mathbf{k}) = \prod_{\tau \in \mathbf{a}} \frac{r_{k_\tau}}{s_{k_\tau}}$, where for $k \in \mathbb{Z}_{\geq 0}$ we denote

$$r_k = \frac{2^{2k+1} (k!)^2}{(2k)!},$$

and s_k is given by (2.10).

Proof. Follows from Corollary 3.7, Theorem 3.8 and the next lemma. \square

Lemma 3.13. *Assume that $V = V_{\mathbf{k}}$ as in Section 2, and let $P_D \in V_{\mathbf{k}}$ as in (2.6). Then*

$$\langle\langle \cdot, \cdot \rangle\rangle = \frac{c(\mathbf{k})}{D^{\mathbf{k}}} \langle \cdot, \cdot \rangle_{P_D}.$$

Proof. Let $M_\infty : B_\infty^\times / F_\infty^\times \rightarrow \mathbb{R}$ denote the multiplicity function considered in [Xue06, (3.9)]. Note that M_{P_D} factors through $B_\infty^\times / F_\infty^\times$, since the representation $(\rho_{\mathbf{k}}, V_{\mathbf{k}})$ does. Furthermore, M_{P_D} and M_∞ are, locally, the matrix coefficient of the (up to multiplication by scalars) unique vector in V_{k_τ} fixed by the action of $K_\tau^\times / F_\tau^\times$ —the first claim follows by definition; for the second, see [Xue06, Lemma 3.13]. This implies $M_\infty = \frac{M_\infty(1)}{M_{P_D}(1)} M_{P_D}$.

Since $\langle\langle \cdot, \cdot \rangle\rangle$ is defined in the same fashion as $\langle \cdot, \cdot \rangle_{P_D}$ but using M_∞ instead of M_{P_D} , we have that

$$\langle\langle \cdot, \cdot \rangle\rangle = \frac{M_\infty(1)}{M_{P_D}(1)} \langle \cdot, \cdot \rangle_{P_D}.$$

Since $M_\infty(1) = \prod_{\tau \in \mathbf{a}} r_{k_\tau}$ and $M_{P_D}(1) = \langle P_D, P_D \rangle$, this together with (2.9) completes the proof. \square

4. A RESULT FOR CERTAIN ORDERS

Assume in this section that $R \subseteq B$ is an order of discriminant \mathfrak{N} satisfying that for every $\mathfrak{p} \mid \mathfrak{N}$ the Eichler invariant $e(R_{\mathfrak{p}})$ is not zero. If $e(R_{\mathfrak{p}}) = 1$ then

$$(4.1) \quad R_{\mathfrak{p}} \simeq \left\{ \begin{pmatrix} a & b \\ \pi_{\mathfrak{p}}^r c & d \end{pmatrix} : a, b, c, d \in \mathcal{O}_{F_{\mathfrak{p}}} \right\},$$

where $r = v_{\mathfrak{p}}(\mathfrak{N})$. If $e(R_{\mathfrak{p}}) = -1$ and we let L be the unique unramified quadratic extension of $F_{\mathfrak{p}}$, then

$$(4.2) \quad R_{\mathfrak{p}} \simeq \left\{ \begin{pmatrix} a & \pi_{\mathfrak{p}}^r b \\ \pi_{\mathfrak{p}}^{r+t} \bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathcal{O}_L \right\},$$

where $t \in \{0, 1\}$ and $2r + t = v_{\mathfrak{p}}(\mathfrak{N})$.

Proposition 4.3. *Let \mathfrak{p} be a prime ideal of F , and let $\text{Bil}(R_{\mathfrak{p}}) = R_{\mathfrak{p}}^{\times} \backslash N(R_{\mathfrak{p}}) / F_{\mathfrak{p}}^{\times}$.*

- (1) *If $\mathfrak{p} \nmid \mathfrak{N}$, then $\text{Bil}(R_{\mathfrak{p}})$ is the trivial group.*
- (2) *If $\mathfrak{p} \mid \mathfrak{N}$, then $\text{Bil}(R_{\mathfrak{p}})$ is a group of order two generated by the equivalence class of an element $w_{\mathfrak{p}} \in R_{\mathfrak{p}} \cap N(R_{\mathfrak{p}})$ which, in terms of the identifications given by (4.1) and (4.2), is given by*

$$w_{\mathfrak{p}} = \begin{cases} \begin{pmatrix} 0 & 1 \\ \pi_{\mathfrak{p}}^r & 0 \end{pmatrix}, & \text{if } e(R_{\mathfrak{p}}) = 1, \\ \begin{pmatrix} 0 & \pi_{\mathfrak{p}}^r \\ \pi_{\mathfrak{p}}^{r+t} & 0 \end{pmatrix}, & \text{if } e(R_{\mathfrak{p}}) = -1. \end{cases}$$

Proof.

- (1) See [Vig80, II.§4, Théorème 2.3].
- (2) See [Hij74, (2.2)] and [Piz76, Proposition 3] for the cases $e(R_{\mathfrak{p}}) = 1$ and $e(R_{\mathfrak{p}}) = -1$ respectively. In the latter the author considers the case when $t = 1$, but the proof is valid in the general case. \square

From these local facts and (1.1) we get the following statement.

Proposition 4.4. *The group $\text{Bil}(R)$ is isomorphic to $\prod_{\mathfrak{p} \mid \mathfrak{N}} \mathbb{Z}/2\mathbb{Z}$, and $\widetilde{\text{Bil}}(R)$ is a finite group of order $h_F 2^{\omega(\mathfrak{N})}$.*

Let $D \in F^+$. Let $K = F(\sqrt{-D})$. By Proposition 2.11 there exists a unique $\mathfrak{a} \in \mathcal{J}_F$ such that $(-D, \mathfrak{a})$ is a fundamental discriminant. Since \mathfrak{a} is determined by D , we omit it in the subindexes for the rest of this section.

As in Section 3, we assume that there exists an embedding $K \hookrightarrow B$ such that $\mathcal{O}_K \subseteq R$, i.e. such that $1 \in \widetilde{X}_D$. There is a left action of $\widetilde{\text{Bil}}(R)$ on X_D , induced by the action of $N(\widehat{R})$ on \widetilde{X}_D by left multiplication. There is a right action of $\text{Cl}(K) = \widehat{\mathcal{O}}_K^{\times} \backslash \widehat{K}^{\times} / K^{\times}$ on X_D , induced by the action of \widehat{K}^{\times} on \widetilde{X}_D by right multiplication.

Lemma 4.5. *Let $X_{D,\mathfrak{p}} = \{x_{\mathfrak{p}} \in B_{\mathfrak{p}}^{\times} : K_{\mathfrak{p}} \cap x_{\mathfrak{p}}^{-1} R_{\mathfrak{p}} x_{\mathfrak{p}} = \mathcal{O}_{K_{\mathfrak{p}}}\}$.*

- (1) *The action of $\mathcal{O}_{K_{\mathfrak{p}}}^{\times} \backslash K_{\mathfrak{p}}^{\times}$ on $R_{\mathfrak{p}}^{\times} \backslash X_{D,\mathfrak{p}}$ is free.*
- (2) *$X_{D,\mathfrak{p}} = N(R_{\mathfrak{p}}) K_{\mathfrak{p}}^{\times}$.*

Proof.

- (1) Let $a_{\mathfrak{p}} \in K_{\mathfrak{p}}^{\times}$ and $x_{\mathfrak{p}} \in X_{D,\mathfrak{p}}$ be such that there exists $u_{\mathfrak{p}} \in R_{\mathfrak{p}}^{\times}$ with $x_{\mathfrak{p}} a_{\mathfrak{p}} = u_{\mathfrak{p}} x_{\mathfrak{p}}$. Then $a_{\mathfrak{p}} = x_{\mathfrak{p}}^{-1} u_{\mathfrak{p}} x_{\mathfrak{p}} \in K_{\mathfrak{p}} \cap x_{\mathfrak{p}}^{-1} R_{\mathfrak{p}} x_{\mathfrak{p}} = \mathcal{O}_{K_{\mathfrak{p}}}^{\times}$.
- (2) Given $x_{\mathfrak{p}} \in X_{D,\mathfrak{p}}$, let $Q_{\mathfrak{p}} = x_{\mathfrak{p}}^{-1} R_{\mathfrak{p}} x_{\mathfrak{p}}$. Since $R_{\mathfrak{p}}$ and $Q_{\mathfrak{p}}$ contain $\mathcal{O}_{K_{\mathfrak{p}}}$ and have the same discriminant, by [Gro88, Proposition 3.4] there exists $a_{\mathfrak{p}} \in K_{\mathfrak{p}}^{\times}$ such that $a_{\mathfrak{p}}^{-1} R_{\mathfrak{p}} a_{\mathfrak{p}} = Q_{\mathfrak{p}}$. Then $x_{\mathfrak{p}} \in N(R_{\mathfrak{p}}) a_{\mathfrak{p}}$. \square

Lemma 4.6. *Let $\mathfrak{p} \mid \mathfrak{N}$. Let $w_{\mathfrak{p}} \in N(R_{\mathfrak{p}})$ be as in Proposition 4.3. If $w_{\mathfrak{p}} \in R_{\mathfrak{p}}^{\times} K_{\mathfrak{p}}^{\times}$, then the extension $K_{\mathfrak{p}}/F_{\mathfrak{p}}$ is ramified.*

Proof. Write $w_{\mathfrak{p}} = u_{\mathfrak{p}} a_{\mathfrak{p}}$ with $u_{\mathfrak{p}} \in R_{\mathfrak{p}}^{\times}$ and $a_{\mathfrak{p}} \in K_{\mathfrak{p}}^{\times}$. Then $a_{\mathfrak{p}} \in \mathcal{O}_{K_{\mathfrak{p}}}$. Using the explicit description of $w_{\mathfrak{p}}$ given in Proposition 4.3 we see that $\pi_{\mathfrak{p}} \nmid a_{\mathfrak{p}}$ in $\mathcal{O}_{K_{\mathfrak{p}}}$. Furthermore, we see that $\pi_{\mathfrak{p}} \mid \mathcal{T}(a_{\mathfrak{p}}), \mathcal{N}(a_{\mathfrak{p}})$ in $\mathcal{O}_{F_{\mathfrak{p}}}$, hence $\pi_{\mathfrak{p}} \mid a_{\mathfrak{p}}^2$ in $\mathcal{O}_{K_{\mathfrak{p}}}$. Thus $\pi_{\mathfrak{p}}$ is ramified in $K_{\mathfrak{p}}$. \square

As a consequence of these lemmas and Proposition 4.3 we obtain the following result.

Proposition 4.7. *The group $\mathrm{Cl}(K)$ acts freely on X_D , and the action of $\mathrm{Bil}(R)$ on $X_D/\mathrm{Cl}(K)$ is transitive. Furthermore, the latter action is free if $(\mathfrak{D}_K : \mathfrak{N}) = 1$.*

Let $\eta_D \in \mathcal{M}_{\mathbf{k}}(R)$ be as in (2.13) and let $\psi_{P_D} \in \mathcal{M}_{\mathbf{k}}(R, \mathbb{1})$ be as in (3.6). We conclude this section by relating these quaternionic modular forms.

Proposition 4.8. *Assume that $(\mathfrak{D}_K : \mathfrak{N}) = 1$. Then*

$$\eta_D = \sum_{z \in \mathrm{Bil}(R)} \psi_{P_D} \cdot z.$$

In particular, $\eta_D \in \mathcal{M}_{\mathbf{k}}(R, \mathbb{1})^{\mathrm{Bil}(R)}$.

Proof. Since $1 \in \tilde{X}_D$, using (1.3) and Proposition 4.7 we get that

$$\sum_{z \in \mathrm{Bil}(R)} \psi_{P_D} \cdot z = \frac{1}{t_K} \sum_{z \in \mathrm{Bil}(R)} \sum_{a \in \mathrm{Cl}(K)} \varphi_{z^{-1}a, P_D} = \frac{1}{t_K} \sum_{x \in X_D} \varphi_{x, P_D} = \eta_D.$$

□

The following statement follows from this result and Proposition 4.4.

Corollary 4.9. *Assume that $(\mathfrak{D}_K : \mathfrak{N}) = 1$. If $\varphi \in \mathcal{M}_{\mathbf{k}}(R, \mathbb{1})^{\mathrm{Bil}(R)}$, then*

$$\langle \varphi, \eta_D \rangle = 2^{\omega(\mathfrak{N})} \langle \varphi, \psi_{P_D} \rangle.$$

5. MAIN THEOREM

Let $\mathbf{k} \in \mathbb{Z}_{\geq 0}^{\mathbf{a}}$, let $\mathfrak{N} \subsetneq \mathcal{O}_F$ be an integral ideal, and let $g \in \mathcal{S}_{2+2\mathbf{k}}(\mathfrak{N}, \mathbb{1})$ be a normalized cuspidal newform with Atkin-Lehner eigenvalues $\varepsilon_g(\mathfrak{p})$ for $\mathfrak{p} \mid \mathfrak{N}$, as in the introduction. Let \mathcal{E} denote the set of functions $\varepsilon : \{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{N}\} \rightarrow \{\pm 1\}$ satisfying

$$(5.1) \quad \varepsilon(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{N})} = \varepsilon_g(\mathfrak{p}) \quad \forall \mathfrak{p} \mid \mathfrak{N}.$$

Note that this set is not empty. This is equivalent to Hypothesis H2.

Given $D \in F^+$ we let $K = F(\sqrt{-D})$, and we denote by χ_D the Hecke character corresponding to the extension K/F . Given $\varepsilon \in \mathcal{E}$ we say that D is of *type* ε when $\chi_D(\mathfrak{p}) = \varepsilon(\mathfrak{p})$ for all $\mathfrak{p} \mid \mathfrak{N}$. In particular \mathfrak{D}_K is prime to \mathfrak{N} . Hypothesis H1 implies that for such D the sign of the functional equation for $L_D(s, g)$ equals 1.

As required by Theorem 3.8, let B be the quaternion algebra over F ramified exactly at the archimedean primes and at \mathcal{W}^- , which is possible by Hypothesis H1. Fix $\varepsilon \in \mathcal{E}$, and let $R = R_{\varepsilon} \subseteq B$ be an order with discriminant \mathfrak{N} and Eichler invariant $e(R_{\mathfrak{p}}) = \varepsilon(\mathfrak{p})$ for every $\mathfrak{p} \mid \mathfrak{N}$. Such order exists by (5.1), and belongs to the class of orders considered in Section 4. Let $\varphi_{\varepsilon} \in \mathcal{S}_{\mathbf{k}}(R, \mathbb{1})$ be a quaternionic modular form in Jacquet-Langlands correspondence with g .

Lemma 5.2. *The quaternionic modular form φ_{ε} is fixed by the action of $\mathrm{Bil}(R)$.*

Proof. Let \mathfrak{p} be a prime dividing \mathfrak{N} , and let $w_{\mathfrak{p}}$ be the generator for $\mathrm{Bil}(R_{\mathfrak{p}})$ given in Proposition 4.4. By the Hecke-linearity and injectivity of the Jacquet-Langlands correspondence, if B is ramified at \mathfrak{p} then $\varphi_{\varepsilon} \cdot w_{\mathfrak{p}} = -\varepsilon_g(\mathfrak{p}) \varphi_{\varepsilon} = \varphi_{\varepsilon}$, and if B is split at \mathfrak{p} then $\varphi_{\varepsilon} \cdot w_{\mathfrak{p}} = \varepsilon_g(\mathfrak{p}) \varphi_{\varepsilon} = \varphi_{\varepsilon}$. Then the result follows, since $\{w_{\mathfrak{p}} : \mathfrak{p} \mid \mathfrak{N}\}$ generates $\mathrm{Bil}(R)$. □

Let c_g the positive real number given by

$$c_g = \langle g, g \rangle \frac{d_F^{1/2}}{h_F} \frac{c(\mathbf{k}) C(\mathfrak{N})}{2^{2\omega(\mathfrak{N})}},$$

where $\langle g, g \rangle$ is the Petersson norm of g , $c(\mathbf{k})$ is as in Corollary 3.12, and $C(\mathfrak{N})$ is as in Theorem 3.8.

Theorem 5.3. *Let $f_\varepsilon = \vartheta(\varphi_\varepsilon) \in \mathcal{S}_{\mathbf{3}/2+\mathbf{k}}^+(4\mathfrak{N}, \chi_1)$. For every $D \in F^+$ of type ε with \mathfrak{D}_K prime to $2\mathfrak{N}$ we have*

$$(5.4) \quad L_D(1/2, g) = c_g \frac{c_D}{D^{\mathbf{k}+1/2}} \frac{|\lambda(D, \mathfrak{a}; f_\varepsilon)|^2}{\langle \varphi_\varepsilon, \varphi_\varepsilon \rangle},$$

where $\mathfrak{a} \in \mathcal{I}_F$ is the unique ideal such that $(-D, \mathfrak{a})$ is a fundamental discriminant, c_D is the positive rational number given by $c_D = m_K^2 \mathcal{N}(\mathfrak{a})$, and $\lambda(D, \mathfrak{a}; f_\varepsilon)$ is the D -th Fourier coefficient of f_ε at the cusp \mathfrak{a} .

Remark 5.5. Hypothesis H1 implies that the sign of the functional equation for $L(s, g)$ equals $(-1)^{\mathbf{k}}$. If Hypothesis H3 does not hold, then both sides of (5.4) vanish trivially, since $\vartheta = 0$ and $L(1/2, g) = 0$. In particular (5.4) still holds, but it can not be used to compute $L(1/2, g \otimes \chi_D)$. This issue will be addressed in a future work by the authors.

Proof. Let T_g be the polynomial in the Hecke operators prime to \mathfrak{N} giving the g -isotypical projection. Let ψ_{P_D} and η_D be as in Corollary 4.9. Since $T_g \psi_{P_D}$ is the φ_ε -isotypical projection of ψ_{P_D} we have that $T_g \psi_{P_D} = \frac{\langle \psi_{P_D}, \varphi_\varepsilon \rangle}{\langle \varphi_\varepsilon, \varphi_\varepsilon \rangle} \varphi_\varepsilon$. Combining this with Proposition 2.14, Corollary 4.9 and Lemma 5.2 we get that

$$\langle T_g \psi_{P_D}, \psi_{P_D} \rangle = \frac{|\langle \psi_{P_D}, \varphi_\varepsilon \rangle|^2}{\langle \varphi_\varepsilon, \varphi_\varepsilon \rangle} = \frac{|\langle \eta_D, \varphi_\varepsilon \rangle|^2}{2^{2\omega(\mathfrak{N})} \langle \varphi_\varepsilon, \varphi_\varepsilon \rangle} = \frac{\mathcal{N}(\mathfrak{a})^2}{2^{2\omega(\mathfrak{N})}} \frac{|\lambda(D, \mathfrak{a}; f_\varepsilon)|^2}{\langle \varphi_\varepsilon, \varphi_\varepsilon \rangle}.$$

Then (5.4) follows from Corollary 3.12. \square

Corollary 5.6. *Under the hypotheses above, assume that $L(1/2, g) \neq 0$. Then $f_\varepsilon \neq 0$ and it maps to g under the Shimura correspondence. Moreover, the set $\{f_\varepsilon : \varepsilon \in \mathcal{E}\}$ is linearly independent.*

In particular, this proves [Sir14, Conjecture 5.6].

Proof. By hypotheses H1 and H3 the sign of the functional equation for $L(s, g)$ equals 1. Hence by [Wal91, Théorème 4] for every $\varepsilon \in \mathcal{E}$ there exists $D_\varepsilon \in F^+$ of type ε with $\mathfrak{D}_\varepsilon = \mathfrak{D}_{F(\sqrt{-D_\varepsilon})}$ odd and prime to \mathfrak{N} such that $L(1/2, g \otimes \chi_{D_\varepsilon}) \neq 0$. Then by (5.4) we have that $\lambda(D_\varepsilon, \mathfrak{a}_\varepsilon; f_\varepsilon) \neq 0$, where $(-D_\varepsilon, \mathfrak{a}_\varepsilon)$ is the discriminant satisfying $D_\varepsilon \mathfrak{a}_\varepsilon^2 = \mathfrak{D}_\varepsilon$. This, together with the Hecke-linearity of the map ϑ , proves the first assertion. The second assertion follows from the fact that if $\varepsilon' \neq \varepsilon$ then $\lambda(D_\varepsilon, \mathfrak{a}_\varepsilon; f_{\varepsilon'}) = 0$. \square

We say that D is *permitted* (with respect to g) if it is of type ε for some $\varepsilon \in \mathcal{E}$.

Corollary 5.7. *Under the hypotheses above, there exists $f \in \mathcal{S}_{\mathbf{3}/2+\mathbf{k}}^+(4\mathfrak{N}, \chi_1)$ whose Fourier coefficients $\lambda(D, \mathfrak{a}; f)$ satisfy*

$$L_D(1/2, g) = \frac{c_D}{D^{\mathbf{k}+1/2}} |\lambda(D, \mathfrak{a}; f)|^2$$

for every permitted D , where $\mathfrak{a} \in \mathcal{I}_F$ is the unique ideal such that $(-D, \mathfrak{a})$ is a fundamental discriminant. Moreover, if $L(1/2, g) \neq 0$, then $f \neq 0$ and it maps to g under the Shimura correspondence.

In particular, this proves [RTV14, Conjecture 2.8].

Proof. This follows from Theorem 5.3 and Corollary 5.6, letting $f = c_g^{1/2} \sum_{\varepsilon \in \mathcal{E}} \frac{f_\varepsilon}{\langle \varphi_\varepsilon, \varphi_\varepsilon \rangle^{1/2}}$. □

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